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Holomorph-convexity of certain covering spaces of projective manifolds

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§1. Motivations and Results.

In this paper we study holomorphically convex properties of covering spaces of compact complex manifolds. A complex space Y is said to be *holomorphically convex* if, given a sequence of distinct points $\{y_n\}_{n\in\mathbb{N}}$ in Y without a limit point in Y, there exists a holomorphic function f on Y such that $\{f(y_n)\}_{n\in\mathbb{N}}$ is unbounded. Obviously a compact complex space is holomorphically convex.

There are two motivations to study such convexity properties. The first one is the following conjecture due to Shafarevich [Shaf, IX.4.3]:

Conjecture. *The universal covering space of a projective manifold is holomorphically convex.*

It is necessary to assume manifolds to be projective, since $\mathbb{C}^2-\{0\}$ is the universal covering of Hopf surface and it is not holomorphically convex by Hartogs' theorem. Kodaira has shown that any compact manifold whose universal covering coincides with $\mathbb{C}^2-\{0\}$ is not algebraic. Furthermore, as a direct consequence of [ABR], $\mathbb{C}^2-\{0\}$ does not cover compact Kähler manifolds. At present we know the following results:

(1.1) Compact Riemann surfaces : Riemann's uniformization theorem.

(1.2) Algebraic surfaces of Kodaira dimension less than 2 by using essentially the Enriques-Kodaira classification of surfaces [Gu], [GS].

(1.3) Algebraic surfaces with enough holomorphic 1-forms (Albanese general type with a certain condition) [Nap].

Key words and phrases. Shafarevich conjecture, holomorphically convex, universal coverings, Stein exhaustions, existence of holomorphic functions.

(1.4) Algebraic surfaces equipped with morphisms to compact Riemann surfaces of genus greater than 1 and all fibers are compact Riemann surfaces of genus greater than 1 [Shab].

(1.5) Manifolds whose universal covering spaces are bounded domains in complex Euclidean spaces [Si].

There are no constructive arguments for the existence of non-constant holomorphic functions on the universal covering spaces in these works and the covering transformations do not have been used there.

The second motivation is the following classical problem in the theory of Stein spaces [GR,V.1.1]: a complex space Y is said to be *Stein* if it is holomorphically convex and *holomorphically separable*, i.e., if x and y are two distinct points of Y, then there is a holomorphic function f on Y such that $f(x) \neq f(y)$.

Exhaustion Problem. *If a domain D of a complex space Y is exhausted by Stein domains i.e. there is a sequence of Stein domains $D_n \subset Y$ such that*

$$D_1 \subset D_2 \subset D_3 \subset \cdots \subset D \text{ and } D = \bigcup_{n=1}^{\infty} D_n,$$

then is D itself Stein ?

In general the answer is negative. There are examples of D, on which no non-constant holomorphic functions exist [FS], [F]. On the other hand, we know some sufficient conditions for D to be Stein. In those cases one of following properties is satisfied:

(1.6) Y has enough non-constant holomorphic functions; in this case the separability is trivial [BS], [M] and [Nar];

(1.7) each pair (D_{n+1}, D_n) has some good relation, so called the Runge pair [St].

In §2 we discuss about the reason why the answer is negative in general.

Now we formulate our result. Let (X, ω) be a compact complex manifold with a Hermitian metric ω, we will use the Hermitian metric on X and the associated fundamental $(1,1)$-form interchangeably. Let (L, h) be a holomorphic line bundle on X with a smooth Hermitian metric h, and denote the curvature form $\Theta := \sqrt{-1}\bar{\partial}\partial \log h$. Let $\pi : \widetilde{X} \longrightarrow X$ be a holomorphic infinite unramified covering from a complex manifold $\widetilde{X}$. We denote the pull back $\widetilde{\omega} := \pi^*\omega, \widetilde{L} := \pi^*L, \widetilde{h} := \pi^*h$ and $\widetilde{\Theta} := \pi^*\Theta$ respectively. Fix an origin $x_0 \in \widetilde{X}$ and denote $d(x_0, x)$ the distance from x_0 to $x \in \widetilde{X}$ with respect to the complete Hermitian metric $\widetilde{\omega}$. Let us assume that there is a non zero holomorphic L^2 section $\tau \in H^0_{(2)}(\widetilde{X}, \widetilde{L}^{\otimes k})$ with respect to $\widetilde{h}$ and $\widetilde{\omega}$ for some $k \in \mathbb{N}$. We can see, by Lemma 3.5 below, that there exists $x_\tau \in \widetilde{X}$ such that $|\tau(x_\tau)| = \max_{x \in \widetilde{X}} |\tau(x)|$, where $|\ \ |$ is the length with respect to $\widetilde{h}$. We call x_τ a *vertex* of τ.

If L is ample, L has a smooth Hermitian metric h of positive curvature and $\widetilde{L}^{\otimes k}$ has enough holomorphic L^2 sections $H^0_{(2)}(\widetilde{X}, \widetilde{L}^{\otimes k})$ for large $k \in \mathbb{N}$ by, for example, L^2-estimate [H]. Furthermore, if $\widetilde{X}$ contains a compact subvariety V of positive dimension, then, obviously, $\widetilde{X}$ is not Stein and $(\tau)_0$, the zero locus of $\tau \in H^0_{(2)}(\widetilde{X}, \widetilde{L}^{\otimes k})$, must intersect with V because $\widetilde{X} - (\tau)_0$ is Stein by Lemma 3.5 below. So $(\tau)_0$ can not go far away from the vertex x_τ, that is, $d(x_\tau, (\tau)_0)$ is bounded by a constant which does not depend on τ. Actually we have

$$d(x_\tau, (\tau)_0) := \inf_{x \in (\tau)_0} d(x_\tau, x) \leq \text{ diam } (X, \omega) + \max_{x \in V} d(x_0, x),$$

where diam (X, ω) is the diameter of X with respect to ω. Note that the above properties and the following assumption $(*)$ do not depend on the choice of the metrics h, ω and x_0. Our main results are to show that the existence of non-constant holomorphic functions on $\widetilde{X}$ under the assumption which, in a certain sense, is opposite to what we talked about above.

Theorem 1. *Let*

$$\begin{array}{ccc} (\widetilde{L}, \widetilde{h}) & \longrightarrow & (L, h) \\ \downarrow & & \downarrow \\ (\widetilde{X}, \widetilde{\omega}) & \xrightarrow{\pi} & (X, \omega) \end{array}$$

,$x_0 \in \widetilde{X}$ and $d(x_0,\ \ )$ be as above, but not necessarily L is ample. Assume that

$$(*) \qquad \varlimsup_{k \to +\infty} \sup\{d(x_\tau, (\tau)_0) \mid \tau \in H^0_{(2)}(\widetilde{X}, \widetilde{L}^{\otimes k})\} = +\infty.$$

Then $\widetilde{L}$ admits a flat Hermitian structure, that is, $\widetilde{L}$ is given by a representation $\pi_1(\widetilde{X}) \longrightarrow S^1$ in the unit circle. In particular, $\widetilde{L} \cong \mathcal{O}_{\widetilde{X}}$ if $\widetilde{X}$ is the universal cover.

We have existence theorems under a condition that L has a weak positivity in some sense.

Theorem 2. *Assume that $\widetilde{X}$ is the universal cover, $\kappa(L) > 0$ and $(*)$. Then there exists a non-constant holomorphic function on $\widetilde{X}$.*

Where $\kappa(L)$ stands for the Iitaka-Kodaira dimension of L, i.e.,

$$\kappa(L) := \begin{cases} -\infty & \text{if } H^0(X, L^{\otimes\nu}) = 0 \text{ for any } \nu \in \mathbb{N}, \\ \max\{\ k \in \mathbb{N} \cup \{0\} \mid \overline{\lim}_{\nu\to+\infty} \nu^{-k} \dim H^0(X, L^{\otimes\nu}) > 0\ \} & \text{otherwise.} \end{cases}$$

We also have convexity for positive L.

Theorem 3. *Assume that L is ample and $(*)$. Then $\widetilde{X}$ is Stein and $\widetilde{L}$ is torsion, i.e., there exists $m \in \mathbb{N}$ such that $\widetilde{L}^{\otimes m} \cong \mathcal{O}_{\widetilde{X}}$.*

In Theorem 3, by the assumption $(*)$, we will see that $\widetilde{X}$ has a Stein exhaustion (§3. (II)), but the Stein exhaustion does not have desirable property like (1.7). The key is to reconstruct "nice" Stein exhaustion by using the covering transformation effectively (Proposition 3.7).

In [St], holomorphic functions were constructed by approximation which is a higher dimensional analogy of classical Runge's approximation theorem.

On the other hand, our proof will be done by constructing certain plurisubharmonic function based on the theorem of Grauert [G], Narasimhan [Nar] and Hörmander [H] (Lemma 2.2).

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§2. Stein spaces and Stein exhaustions.

2.1. Stein spaces.

Let Y be a complex space with a countable topology. An upper semi-continuous function $\varphi : Y \to [-\infty, +\infty)$ is said to be *plurisubhamonic*, psh for short, if for every holomorphic mapping $\Phi : \Delta \to Y$ from the unit disk $\Delta \subset \mathbb{C}$, $\varphi \circ \Phi$ is subhamonic on Δ. φ is said to be *strictly* psh if for every smooth real valued function λ with compact support in Y, $\varphi + \varepsilon\lambda$ is psh for every sufficiently small positive real number ε. If Y is a complex manifold and $\varphi : Y \to \mathbb{R}$ is a function of class C^2, the *Levi form* of φ is the Hermitian $(1,1)$-form defined by

$$(\sqrt{-1}\partial\bar{\partial}\varphi)(y) := \sqrt{-1}\sum \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}(y) dz^i \wedge d\bar{z}^j$$

for $y \in Y$ and for any local coordinate system $(z^1, \dots, z^n)$ around y. φ is strictly psh (resp. psh) if and only if its Levi form is positive (resp. positive semi-) definite for every $y \in Y$.

A real valued function φ on Y is said to be an *exhaustion* function if for every $c \in \mathbb{R}$

$$Y_c := \{y \in Y \mid \varphi(y) < c\}$$

is a relatively compact open subset of Y. The following is the solution to the Levi problem due to Grauert [G], Narasimhan [Nar] and Hörmander [H].

Lemma 2.2. *A complex space is Stein if and only if it has a continuous strictly psh exhaustion function. Moreover, on a Stein space, the strictly psh exhaustion function may be chosen real analytic.*

2.3. Stein exhaustoins.

Let Y be a Stein space with a smooth strictly psh exhaustion function φ. For every $c \in \mathbb{R}$, $Y_c := \{y \in Y \mid \varphi(y) < c\} \Subset Y$ is Stein with a smooth strictly psh exhaustion function $-\log(c - \varphi)$. Hence Y is exhausted by (non-trivial) Stein domains, $Y_n := \{y \in Y \mid \varphi(y) < c_n\}$ with a smooth strictly psh exhaustion function $\varphi_n := -\log(c_n - \varphi)$,

$$(2.4) \qquad (Y_1, \varphi_1) \Subset (Y_2, \varphi_2) \Subset (Y_3, \varphi_3) \Subset \cdots \Subset (Y, \varphi) \text{ and } Y = \bigcup_{n=1}^{\infty} Y_n,$$

for appropriate choice of real numbers $c_1 < c_2 < c_3 < \cdots$.

Consider a complex space Z, not necessarily Stein itself, which is exhausted by Stein domains Z_n with a smooth strictly psh exhaustion function ψ_n

$$(2.5) \qquad (Z_1, \psi_1) \Subset (Z_2, \psi_2) \Subset (Z_3, \psi_3) \Subset \cdots \Subset Z \text{ and } Z = \bigcup_{n=1}^{\infty} Z_n.$$

Note that for any Stein exhaustion, we can construct a Stein exhaustion each open set is connected and relatively compact. The critical difference between (2.4) and (2.5) is whether the strictly psh exhaustion functions on Y_n and Z_n are obtained from only one function or not. For $\{(Y_n, \varphi_n)\}_{n \in \mathrm{N}}$, by modification we get $\{\varphi'_n\}_{n \in \mathrm{N}}$, where $\varphi'_n := c_n - \exp(-\varphi_n) \in C^\infty(Y_n)$. The limit function exists and define a smooth strictly psh exhaustion function φ on Y, and recover that Y is Stein. On the other hand, consider a sequence of functions $\{\psi'_n\}_{n\in\mathrm{N}}$, $\psi'_n \in C^\infty(Z_n)$, obtained by certain processes from $\{\psi_n\}_{n\in\mathrm{N}}$. If we want to show that Z is Stein by using Lemma 2.2, the following troubles may occur:

(2.6) (any subsequence of) $\{\psi'_n\}_{n\in\mathrm{N}}$ does not convergent — their vertexies are unbounded, cf. §3 III.

(2.7) $\{\psi'_n\}_{n\in\mathrm{N}}$ convergents, but the limit function is not strictly psh — positivity of $\sqrt{-1}\partial\bar{\partial}\psi'_n$ are not uniform, cf. §3 II.

(2.8) $\{\psi'_n\}_{n\in\mathrm{N}}$ convergents, but the limit is not an exhaustion function — geometry of Z at infinity is uncertain, cf. Remark 3.4 (1).

§3. Construction of nice (Stein) exhaustions.

Let

$$\begin{array}{ccc}(\widetilde{L},\widetilde{h}) & \longrightarrow & (L,h)\\ \downarrow & & \downarrow\\ (\widetilde{X},\widetilde{\omega}) & \xrightarrow{\pi} & (X,\omega)\end{array}$$

,$x_0 \in \widetilde{X}$ and $d(x_0, \ )$ with the assumption $(*)$ be in §1. Fix a fundamental domain $X_0 \ni x_0$ of the covering π, X_0 is a bounded domain because X is compact. Let G be the covering transformation group of π. There are three important ingradients to construct nice (Stein) exhaustions.

(I) Bounded geometry.

$(\widetilde{X},\widetilde{\omega})$ is not compact, but it has some nice properties like compact manifolds. Recall some definitions [CY], [Kob].

Definition 3.1. Let V be an open set in $\mathbb{C}^n$. A holomorphic map from V into a complex manifold M of dimension n is called a *quasi-coordinate map* if it is of maximal rank everywhere in V. The pair $(V\ ;\ (v^1,\cdots,v^n))$ the Euclidean coordinates of $\mathbb{C}^n$ is a *local quasi-coordinate* of M

Definition 3.2. A complete Hermitian manifold (M,g) is said to be have *bounded geometry* if there exist a positive real number $1/2 < R < 1$ and a family of local quasi-coordinates $\mathcal{V} = \{(V;(v^1,\cdots,v^n))\}$ of M with the following properties.

(i) M is covered by the images of $(V;(v^1,\cdots,v^n))$'s.

(ii) Each V, as an open subset of $\mathbb{C}^n$, contains a ball of radius R.

(iii) There eixst positive constants C and A_l ($l = 0, 1, 2, \cdots$) independent of V's such that at each $(V; (v^1, \cdots, v^n))$, the inequalities

$$C^{-1}(\delta_{ij}) < (g_{i\bar{j}}) < C(\delta_{ij}),$$

$$\left| \frac{\partial^{|p|+|q|} g_{i\bar{j}}}{\partial v^p \partial \bar{v}^q} \right| < A_{|p|+|q|}, \text{ for any multi-indices } p \text{ and } q,$$

hold, where $g_{i\bar{j}}$ denote the components of g with respect to v^i's. The constants R, C and A_l ($l = 0, 1, 2, \cdots$) are refered as the *constants associated with the bounded geometry of* (M, g).

$(\widetilde{X}, \widetilde{\omega})$ has bounded geometry, since $\widetilde{X}$ covers the compact Hermitian manifold (X, ω) and $\widetilde{\omega} = \pi^* \omega$. By smoothing the distance function $d(x_0, \ )$, we have

Lemma 3.3 ([Nap, Lemma 3.2]). *There exist a positive constant c_0 depending only on the constants associated with the bounded geometry of $(\widetilde{X}, \widetilde{\omega})$ and a smooth positive function r on $\widetilde{X}$ such that*

(i) $c_0 d(x_0, x) \leq r(x) \leq c_0(d(x_0, x) + 1)$ for any $x \in \widetilde{X}$,

(ii) $|(\partial + \bar{\partial})\, r|_{\widetilde{\omega}^*} \leq c_0$ on $\widetilde{X}$ and

(iii) $-c_0 \widetilde{\omega} \leq \sqrt{-1}\partial\bar{\partial}\, r \leq c_0 \widetilde{\omega}$ on $\widetilde{X}$.

Remark 3.4. (1) The existence of a function with properties (i), (ii) and (iii) on a Hermitian manifold strongly restricts the geometry of it.

(2) The above lemma can be formulated for any Hermitian manifold (M, g) with bounded geometry [Nap].

(II) Uniform positivity.

The following lemma is important to construct relatively compact Stein subdomains and to guarantee the limit function corresponding to (2.7) is strictly psh.

Lemma 3.5. *For any non zero $\tau \in H^0_{(2)}(\widetilde{X}, \widetilde{L}^{\otimes k})$, $\widetilde{X} - (\tau)_0$ is exhausted by a smooth function $-\log|\tau|^2$, where $|\tau|$ is the length of τ with respect to the Hermitian metric $\widetilde{h}^{\otimes k}$, and $\sqrt{-1}\partial\bar{\partial}(-k^{-1}\log|\tau|^2) = \widetilde{\Theta}$. In particular, $\widetilde{X} - (\tau)_0$ is Stein if L is ample.*

Proof. By definitions of $\widetilde{h}$ and $\widetilde{\Theta}$, we have $\sqrt{-1}\partial\bar{\partial}(-k^{-1}\log|\tau|^2) = \widetilde{\Theta}$. All we have to show is that $-\log|\tau|^2$ is an exhaustion function of $\widetilde{X} - (\tau)_0$. Let $j : V \longrightarrow \widetilde{X}$

be a local quasi-coordinate centered at $x \in \widetilde{X}$. By the mean value inequality for subharmonic functions, we have

$$|\tau(x)|^2 \leq \frac{c}{\mathrm{vol} B_R} \int_{j(B_R)} |\tau|^2 d\widetilde{V},$$

where c is a positive constant depending only on the constants associated with the bounded geometry of $(\widetilde{X}, \widetilde{\omega})$, not depend on x, $B_R \subset V$ is the ball of radius R, vol B_R is the volume with respect to the standard Euclidean metric (Definition 3.2 (ii)) and where $d\widetilde{V}$ is the volume element determined from $\widetilde{\omega}$. For any $\varepsilon > 0$, there exist constants $d_1 > d_0 > 0$ such that

$$\int_{B_0} |\tau|^2 d\widetilde{V} \geq \int_{\widetilde{X}} |\tau|^2 d\widetilde{V} - \varepsilon \quad \text{and}$$
$$j(B_R) \cap B_0 = \emptyset \quad \text{for any } x \in \widetilde{X} - B_1,$$

where $B_i = \{x \in \widetilde{X} \mid d(x_0, x) \leq d_i\}$ for $i = 0, 1$. Then

$$|\tau(x)|^2 \leq \frac{c}{\mathrm{vol} B_R} \int_{j(B_R)} |\tau|^2 d\widetilde{V} \leq \frac{c}{\mathrm{vol} B_R} \varepsilon$$

for any $x \in \widetilde{X} - B_1$. Since c and vol B_R depend only on the constants associated with the bounded geometry of $(\widetilde{X}, \widetilde{\omega})$, $|\tau(x)| \to 0$ as $d(x_0, x) \to +\infty$ and this means $-\log |\tau|^2$ is an exhaustion function. ■

(III) Group action.

We do not use the assumption $(*)$ and the covering transformation yet. The former is utilized to construct a (Stein) exhaustion, the latter controls the (Stein) exhaustion and guarantees the existence of the limit function corresponding to (2.6).

Let $\tau_1 \in H^0_{(2)}(\widetilde{X}, \widetilde{L}^{\otimes k_1})$ be a non-zero section for some $k_1 \in \mathbb{N}$. For a suitable choice of $c_1 \in \mathbb{R}$, the following X_1 is non-empty.

$$X_1 := \text{ a connected component of } \left\{ x \in \widetilde{X} \mid \varphi_1(x) := -\frac{1}{k_1} \log |\tau_1|^2 < c_1 \right\}$$

which contains a vertex x_1 of τ_1 (see §1). We can see that X_1 is a relatively compact domain of $\widetilde{X}$ by Lemma 3.5. By the definition of vertices, $\varphi_1(x_1) = \inf_{X_1} \varphi_1$. Furthermore we may assume that $\varphi_1(x_1) = 0$ by multiplying some constant to τ.

Now there exists a covering transformation $g \in G$ such that $g^{-1}(x_1) \in \overline{X_0}$: the closure of the fundamental domain. By regarding $g^*\tau_1$ as τ_1, we have

$$x_0 \in X_0 \Subset X_1 \Subset \widetilde{X} \text{ and } x_1 \in \overline{X_0}$$

for suitable choices of $k_1 \in \mathbb{N}$, $\tau_1 \in H^0_{(2)}(\widetilde{X}, \widetilde{L}^{\otimes k_1})$ and $c_1 \in \mathbb{R}$ by $(*)$. Iterating these processes, we have:

Proposition 3.6. *For every $n \in \mathbb{N}$, there exist $k_n \in \mathbb{N}, \tau_n \in H^0_{(2)}(\widetilde{X}, \widetilde{L}^{\otimes k_n}), c_n \in \mathbb{R}$ and*

$$X_n := \text{ a connected component of } \left\{ x \in \widetilde{X} \mid \varphi_n(x) := -\frac{1}{k_n} \log |\tau_n|^2 < c_n \right\}$$

which contains a vertex x_n of τ_n, such that $\{X_n\}_{n \in \mathbb{N}}$ defines an exhaustion

$$x_0 \in X_0 \Subset X_1 \Subset X_2 \Subset \cdots \Subset \widetilde{X}, \qquad \widetilde{X} = \bigcup_{n=1}^{\infty} X_n, \tag{3.7}$$

and $x_n \in \overline{X_0}$ for any $n \in \mathbb{N}$. If L is ample, (3.7) is a Stein exhaustion of $\widetilde{X}$.

§4. Proof of Theorem.

We use the same notations in Proposition 3.6.

4.1. Proof of Theorem 1.

For every $m \in \mathbb{N}$, consider a sequence

$$\{\varphi_{n,m} - \varphi_{m,m} + c_m\}_{n \geq m}, \quad \text{where} \quad \varphi_{n,m} := \varphi_n|_{X_m} \in C^\infty(\overline{X_m}).$$

Since $\varphi_{n,m} \geq 0$ and $\varphi_{m,m} - c_m \leq 0$ on X_m,

$$\varphi_{n,m} - \varphi_{m,m} + c_m \geq 0 \quad \text{and}$$
$$\sqrt{-1}\partial\bar{\partial}(\varphi_{n,m} - \varphi_{m,m} + c_m) = 0 \quad \text{on } X_m$$

for any $n \geq m$. Take a domain X'_m such that $X_{m-1} \Subset X'_m \Subset X_m$. By the Harnack's inequality [GT], using a family of local quasi-coordinates of $(\widetilde{X}, \widetilde{\omega})$, there exists a constant C_m depending only on X_m, X'_m and the constants associated with the bounded geometry of $(\widetilde{X}, \widetilde{\omega})$ such that

$$\begin{aligned}
\sup_{X'_m}(\varphi_{n,m} - \varphi_{m,m} + c_m) &\leq C_m \inf_{X'_m}(\varphi_{n,m} - \varphi_{m,m} + c_m) \\
&\leq C_m(\varphi_{n,m}(x_n) - \varphi_{m,m}(x_n) + c_m) \\
&\leq C_m c_m
\end{aligned}$$

for any $n \geq m$. Hence we have

$$\sup_{X'_m} |\varphi_{n,m} - \varphi_{m,m}| \leq c_m(1 + C_m)$$

for any $n \geq m$. Thus we have a bounded sequence of smooth functions on X'_m

$$\{(\varphi_{n,m} - \varphi_{m,m})|_{X'_m}\}_{n \geq m} \quad \text{such that} \quad \sqrt{-1}\partial\bar{\partial}(\varphi_{n,m} - \varphi_{m,m})|_{X'_m} = 0.$$

Hence by the Harnack's convergence theorem [GT], $\{(\varphi_{n,m} - \varphi_{m,m})|_{X'_m}\}_{n \geq m}$ contains a subsequence $\{(\varphi_{n(m),m} - \varphi_{m,m})|_{X'_m}\}$ converges uniformly on the relatively compact subdomain X_{m-1} to a function

$$\Phi_{m-1} := \lim_{n(m) \to +\infty} (\varphi_{n(m),m} - \varphi_{m,m}) \in C^{\infty}(X_{m-1}) \quad \text{such that} \quad \sqrt{-1}\partial\bar{\partial}\Phi_{m-1} = 0.$$

Thus we have $\{\Phi_{m-1}\}_{m \geq 1}$ step by step from $m = 1$. We define $\Phi(x) := \Phi_{m-1}(x) + \varphi_m(x)$ if $x \in X_{m-1}$, note that indices are renumbered, then Φ is well defined and a smooth $\mathbb{R}_{\geq 0}$-valued function on $\widetilde{X}$ with $\sqrt{-1}\partial\bar{\partial}\Phi = \widetilde{\Theta}$. Since $\sqrt{-1}\bar{\partial}\partial \log(e^{\Phi}\widetilde{h}) = -\sqrt{-1}\partial\bar{\partial}\Phi + \widetilde{\Theta} = 0$, $e^{\Phi}\widetilde{h}$ is a Hermitian flat metric on $\widetilde{L}$ ∎

4.2. Proof of Theorem 2.

By Theorem 1, $\widetilde{L} \cong \mathcal{O}_{\widetilde{X}}$. Since $\kappa(L) > 0$, there exists a non zero holomorphic section $f \in H^0(X, L^{\otimes \nu})$ for some $\nu \in \mathbb{N}$ with non empty zero locus $(f)_0$. Then the pull back $\pi^* f \in H^0(\widetilde{X}, \widetilde{L}^{\otimes \nu})$ corresponds to a non-constant holomorphic function on $\widetilde{X}$. ∎

4.3. Proof of Theorem 3.

If L is ample, we may take $\omega = \Theta$ as a Kähler metric on X. Let us consider a function $\Phi + (2c_0)^{-1}r$ on $\widetilde{X}$, where Φ is the potential function as in 4.1, and where c_0 and r are the functions in Lemma 3.3. Φ is a smooth semi-positive valued function with $\sqrt{-1}\partial\bar{\partial}\Phi = \widetilde{\Theta}$. r is a smooth exhaustion function like the distance function from x_0 with $\sqrt{-1}\partial\bar{\partial}r \geq -c_0\widetilde{\Theta}$. Hence $\Phi + (2c_0)^{-1}r$ is a smooth strictly psh exhaustion function on $\widetilde{X}$. Hence $\widetilde{X}$ is Stein by Lemma 2.2.

We consider the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}^*_{\widetilde{X}} \longrightarrow 0,$$

and a part of the long exact sequence,

$$H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \longrightarrow \mathrm{Pic}\widetilde{X} \xrightarrow{\delta} H^2(\widetilde{X}, \mathbb{Z}) \longrightarrow H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}),$$

where δ is the connection homomorphism. Since $\widetilde{X}$ is Stein, $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$, and we see that δ is an isomorphism. As $\sqrt{-1}\partial\bar{\partial}\Phi = \widetilde{\omega}$, the first Chern class $c_1(\widetilde{L}) = 0$ in $H^2(\widetilde{X}, \mathbb{R})$. We denote j the natural homomorphism $H^2(\widetilde{X}, \mathbb{Z}) \to H^2(\widetilde{X}, \mathbb{R})$. Since $c_1 = j \circ \delta$, $\delta(\widetilde{L}) \in \ker j = H^2(\widetilde{X}, \mathbb{Z})_{\mathrm{tor}}$. This means $\widetilde{L}$ is torsion, and the pull back of $\widetilde{L}$ is trivial after taking certain finite étale covering of $\widetilde{X}$. ∎

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